Direct and Indirection Proportion

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Definitions and Intuition

Direct Proportion

Direct proportion describes two amounts where one amount increases at the **same rate** as the other amount

For example:
Dan is paid £11 per hour. As the number of hours increases, the amount he is paid increases at the **same rate**

- If Dan works for 2 hours he is paid $11 \times 2 = £22$
- If Dan works for 3 hours he is paid $11 \times 3 = £33$

Notice the **multiplication** here

As a formula we can write this as

Money Earnt = $11 \times$ number of hours worked

More generally we can write this formula as

Money Earnt = some fixed constant $\times$ number of hours worked

Tip for later: You will see below we write this 'fixed constant' generally as k

Indirect Proportion

Indirect proportion describes two amounts where if one amount increases the other amount decreases and if one amount decreases, the other amount increases

For example:
It takes 1 worker, 9 hours to dig a hole. As the number of workers **increases**, the number of hours it takes to dig the same hole **decreases**.

- 3 workers would take $\frac{9}{3}$ hours = 3 hours
- 9 workers would take $\frac{9}{9}$ hours = 1 hour

Notice the **division** here

As a formula we can write this as

$$\text{Time taken} = \frac{9}{\text{number of workers}}$$

More generally we can write this formula as

$$\text{Time taken} = \frac{\text{some fixed constant}}{\text{number of workers}}$$

Another example of indirect proportion is speed and time. As speed increases it takes less time to cover same distance. As speed decreases, it takes more time to cover the same distance

Pre-Requisite Topics –You need to be good at the following topics first

- Basic Number Substitution (first seen in year 4/5)
- BIDMAS (first seen in year 4/5)
- Changing the subject of a formula (first seen early on in year 10)

Equation Template

In the above examples we have seen that we multiply or divide by some fixed constant. A general formula can be written as below. The pink and blue boxes have been written as a general template case. You will not write the boxes of course. You just need to locate in the question what variable/letters the question wants and replace the boxes. This will make much more sense once you start the examples under the 'grade 5 difficulty type 1' section a bit further down.

Direct Proportion

$\square$ is directly proportional to $\square$

means you can form the following equation

$$\square = k \times \square$$

Take note of how we introduce the constant k with multiplication

'directly proportional' can also be phrased as

- varies directly
- is in proportion to
- varies as
- $\propto$

(if the question doesn't say directly, it implies directly anyway)

Indirect Proportion

$\square$ is indirectly proportional to $\square$

means you can form the following equation

$$\square = \frac{k}{\square}$$

Take note of how we introduce the constant k with division

'Indirectly proportional' can also be phrased as

- varies **inverse**ly
- **inverse**ly proportional

Notice how you need to see the word 'inverse'

Step By Step Method – Use This Method As You Attempt The Examples Below

Step 1: Ask yourself whether you have **direct** or **indirect** proportion in other words $k\times$ or $\frac{k}{}$
Step 2: Write the formula by filling into the template above: Fill the correct variables into $\square$ and $\square$
Step 3: Plug the given values into the variables and solve for k
Step 4: Put k back into your original equation formed in step 2
Step 5: Find whatever variable you're asked to find by substituting the information you're told and re-arranging for the unknown

<u>Tip: Don't lose sight of your goal</u>

- Use the **first bit of information** to **find k** and don't forget what you're doing once you find k. Once you find k **put it straight back** into the **original** equation
- Use the **NEXT bit of information** to **solve** for the **unknown** (your re-arranging of algebra needs to be good)

Grade 5 Difficulty Type 1

Example 1

y is directly proportional to x
When $x = 600,\ y = 10$
i. Find a formula for y in terms of x
ii. Calculate the value of y when $x = 540$

i.
Step 1: Plug into the template

y is directly proportional to x
means
$$y = kx$$

Step 2: Use the first bit of information given to find k (we only use this info once)
$$y = 10, x = 600$$

We plug these values in our equation to find k
$$y = kx$$
$$10 = k(600)$$
$$k = \frac{10}{600} = \frac{1}{60}$$

Step 3: Put the value of k found into the equation
$$y = kx$$
$$y = \frac{1}{60}x$$

ii.
Plug the value given in and solve for the unknown
$$x = 540$$
$$y = \frac{1}{60}x = \frac{1}{60}(540) = 9$$

Hence the value of y when $x = 540$ is **9**

Example 2

m is directly proportion to l^3
When $l = 2, m = 160$
Find the value of m when $l = 3$

Step 1: Plug into the template

m is directly proportional to l^3
means
$$m = kl^3$$

Step 2: Use the first bit of information given to find k (we only use this info once)
$$m = 160 \text{ when } l = 2$$
$$m = kl^3$$
$$160 = k(2)^3$$

We re-arrange to make k the subject
$$k = \frac{160}{8} = 20$$

Step 3: Put the value of k found into the equation
$$m = kl^3$$
$$m = 20l^3$$

Step 4: Plug the other value given in and solve for the unknown

We want m when $l = 3$
$$m = 20l^3$$
$$m = 20(3)^3$$
$$m = 540$$

Hence, the value of m when $l = 3$ is **540**

Example 3

P is inversely proportional to V
When $V = 8, P = 5$
i. Find a formula for P in terms of V
ii. Calculate the value of P when $V = 2$

i.
Step 1: Plug into the template

P is indirectly proportional to V
means
$$P = \frac{k}{V}$$

Step 2: Use the first bit of information given to find k (we only use this info once)
$$P = 5, V = 8$$

We plug these values in our equation to find k
$$P = \frac{k}{V}$$
$$5 = \frac{k}{8}$$
$$k = 5 \times 8 = 40$$

Step 3: Put the value of k found into the equation
$$P = \frac{k}{V}$$
$$P = \frac{40}{V}$$

ii.
Plug the value given in and solve for the unknown
$$V = 2$$
$$P = \frac{40}{V} = \frac{40}{2} = 20$$

Example 4

T is inversely proportional to d^2
$T = 12$ when $d = 8$
Find the value of T when $d = 0.5$

Step 1: Plug into the template

T is indirectly proportional to d^2
means
$$T = \frac{k}{d^2}$$

Step 2: Use the first bit of information given to find k (we only use this info once)
$$T = 12, d = 8$$

We plug these values in our equation to find k
$$T = \frac{k}{d^2}$$
$$12 = \frac{k}{8^2}$$
$$k = 12 \times 64 = 768$$

Step 3: Put the value of k found into the equation
$$T = \frac{k}{d^2}$$
$$T = \frac{768}{d^2}$$

Step 4 : Plug the value given in and solve for the unknown
$$d = 0.5$$
$$T = \frac{768}{d^2} = \frac{768}{(0.5)^2} = \frac{768}{0.25} = 3072$$

Hence the value of T when $d = 0.5$ is **3072**

Example 5

The quantity y varies as the cube of $(x+2)$
$y = 32$ when $x = 0$
Find y when $x = 1$

Step 1: Plug into the template

y varies as the cube of $(x+2)$
y is directly proportional to $(x+2)^3$
means
$$y = k(x+2)^3$$

Step 2: Use the first bit of information given to find k
$$y = 32, x = 0$$

We plug these values in our equation to find k
$$y = k(x+2)^3$$
$$32 = k(0+2)^3$$
$$k = \frac{32}{2^3} = 4$$

Step 3: Put the value of k found into the equation
$$y = k(x+2)^3$$
$$y = 4(x+2)^3$$

Step 4: Plug the other value given in and solve for the unknown
$$x = 1$$
$$y = 4(x+2)^3 = 4(1+2)^3 = 4(27) = 108$$

Example 6

M is proportional to the cube of r
When $r = 3, M = 21.6$
When $r = 5$, find the value of M

Step 1: Plug into the template

M is proportional to the cube of r
M is directly proportional to r^3
means
$$M = kr^3$$

Step 2: Use the first bit of information given to find k
$$M = 21.6, r = 3$$

We plug these values in our equation to find k
$$M = kr^3$$
$$21.6 = k3^3$$
$$k = \frac{21.6}{27} = 0.8$$

Step 3: Put the value of k found into the equation
$$M = kr^3$$
$$M = 0.8r^3$$

Step 4: Plug the other value given in and solve for the unknown
$$r = 5$$
$$M = 0.8r^3 = 0.8(5)^3 = 0.8(125) = 100$$

Grade 5 Difficulty Type 2: Given More Words

Example 1

In a spring, the tension (T newtons) is directly proportional to its extension (x cm)
When the tension is 150 newtons, the extension is 6 cm
i. Find a formula for T in terms of x
ii. Calculate the tension, in newtons, when the extension is 15 cm
iii. Calculate the extension, in cm, when the tension is 600 newtons

i.
Step 1: Plug into the template

T is directly proportional to x
means
$$T = kx$$

Step 2: Use the first bit of information given to find k (we only use this info once)

Tension is 150 newtons when extension is 6cm
$$T = 150, x = 6$$

We plug these values in our equation to find k
$$T = kx$$
$$150 = k(6)$$
$$k = \frac{150}{6} = 25$$

Step 3: Put the value of k found into the equation
$$T = kx$$
$$T = 25x$$

ii.
Plug the value given in and solve for the unknown

Tension is unknown when extension is 15cm
$$x = 15$$
$$T = 25x = 25(15) = 375$$

Hence, when the extension is 15cm, the tension is 375 newtons.

iii.
Plug the value given in and solve for the unknown

Tension is 600 newtons when extension is unknown
$$T = 600$$
$$T = 25x$$
$$600 = 25x$$
$$x = \frac{600}{25} = 24$$

Hence, for the tension to be 600 newtons, the extension is 24cm

Example 2

When cars go round a bend, there is a force F, between the tyres and the ground. F varies directly as the square of the speed, v
When $v = 40, F = 18$
Find F when $v = 32$

Step 1: Plug into the template

y varies as the cube of $(x+2)$
y is directly proportional to $(x+2)^3$
means
$$y = k(x+2)^3$$

Step 2: Use the first bit of information given to find k
$$y = 32, x = 0$$

We plug these values in our equation to find k
$$y = k(x+2)^3$$
$$32 = k(0+2)^3$$
$$k = \frac{32}{2^3} = 4$$

Step 3: Put the value of k found into the equation
$$y = k(x+2)^3$$
$$y = 4(x+2)^3$$

Step 4: Plug the other value given in and solve for the unknown
$$x = 1$$
$$y = 4(x+2)^3 = 4(1+2)^3 = 4(27) = 108$$

Example 3

The force of attraction (F) between two objects is inversely proportional to the square of the distance (d) between them
When $d = 4, F = 30$
Calculate F when $d = 8$

Step 1: Plug into the template

F is indirectly proportional to d^2
means
$$F = \frac{k}{d^2}$$

Step 2: Use the first bit of information given to find k (we only use this info once)
$$F = 30, d = 4$$

We plug these values in our equation to find k
$$F = \frac{k}{d^2}$$
$$30 = \frac{k}{4^2}$$
$$k = 30 \cdot 16 = 480$$

Step 3: Put the value of k found into the equation
$$F = \frac{k}{d^2}$$
$$F = \frac{480}{d^2}$$

Step 4 : Plug the value given in and solve for the unknown
$$d = 8$$
$$F = \frac{480}{d^2} = \frac{480}{(8)^2} = \frac{480}{64} = 7.5$$

Hence the value of F when $d = 8$ is **7.5**

Example 4

The light intensity E, at a surface is inversely proportional to the square of the distance, r, of the surface from the light source
$E = 4$ when $r = 50$
i. Express E in terms of r
ii. Calculate the value of E when $r = 20$
iii. Calculate the value of r when $E = 1600$

i.
Step 1: Plug into the template

E is inversely proportional to the square of r
E is indirectly proportional to r^2
means
$$E = \frac{k}{r^2}$$

Step 2: Use the first bit of information given to find k (we only use this info once)
$$E = 4, r = 50$$

We plug these values in our equation to find k
$$E = \frac{k}{r^2}$$
$$4 = \frac{k}{50^2}$$
$$k = 4 \cdot 2500 = 10000$$

Step 3: Put the value of k found into the equation
$$E = \frac{k}{r^2}$$
$$E = \frac{10000}{r^2}$$

ii.
Plug the value given in and solve for the unknown
$$r = 20$$
$$E = \frac{10000}{r^2} = \frac{10000}{20^2} = \frac{10000}{400} = 25$$

Hence the value of E when $r = 20$ is **25**

iii.
Plug the value given in and solve for the unknown
$$E = 1600$$
$$E = \frac{10000}{r^2}$$
$$1600 = \frac{10000}{r^2}$$
$$r^2 = \frac{10000}{1600} = \frac{100}{16}$$
$$r = \sqrt{\frac{100}{16}} = \frac{10}{4} = 2.5$$

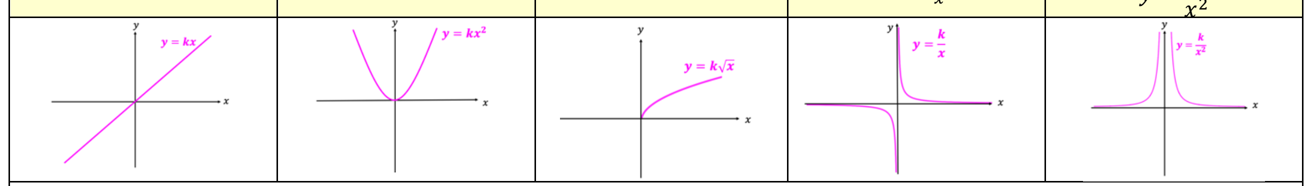
Note : We do not consider the negative value since distance can never be negative

Hence the value of r when $E = 1600$ is **2.5**

Grade 5 Difficulty: Memorising Graphs

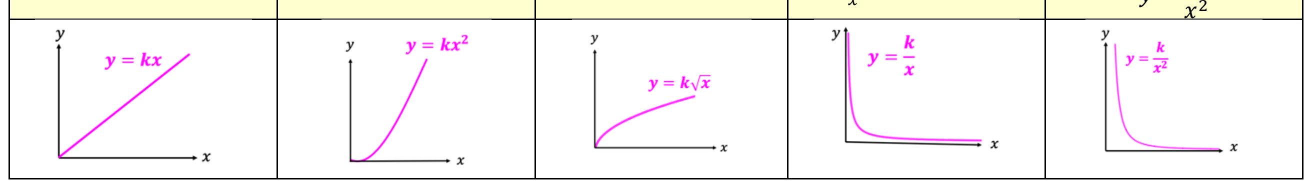
You need to know the basic shapes of proportion graphs. This is purely memorisation of the graph shapes below. You will be given graph shapes and equations and you need to simply match the graphs to the equations.

Direct Proportion – as x increases, y increases			Indirect Proportion as x increases, y decreases	
$y = kx$	$y = kx^2$	$y = k\sqrt{x}$	$y = \frac{k}{x}$	$y = \frac{k}{x^2}$



Note: Sometimes you will see the graph of only the positive x and y region

$y = kx$	$y = kx^2$	$y = k\sqrt{x}$	$y = \frac{k}{x}$	$y = \frac{k}{x^2}$



Grade 6 Difficulty: Given A Table

Example 1

y is directly proportional to $\sqrt{x}$

x	36	a
y	2	5

Work out the value of a

Step 1: Recall the template, and notice that here:

y is directly proportional to $\sqrt{x}$
means
$$y = k\sqrt{x}$$

Step 2: Use the pair of values given in the second column which tell us that (this is just the same as usual except we are not explicitly given the values in words)
$$y = 2 \text{ when } x = 36$$
$$y = k\sqrt{x}$$
$$2 = k(\sqrt{36})$$

We re-arrange to make k the subject
$$k = \frac{2}{6} = \frac{1}{3}$$

Step 3: Put the value of k found into the equation
$$y = k\sqrt{x}$$
$$y = \frac{1}{3}\sqrt{x}$$

Step 4: Plug the other value of y given and find the unknown value
$$x = a \text{ when } y = 5$$
$$y = \frac{1}{3}\sqrt{x}$$
$$5 = \frac{1}{3}\sqrt{a}$$
$$\sqrt{a} = 15$$
$$a = 15^2 = 225$$

Hence, the value of a is **225**

Example 2

x is directly proportional to y

x	60	36	q
y	35	p	17.5

Work out the missing values of p and q in the table

Step 1: Recall the template, and notice that here:

y is directly proportional to x
means
$$y = kx$$

Step 2: Use the pair of values given in the second column
$$y = 35 \text{ when } x = 60$$
$$y = kx$$
$$35 = k(60)$$

We re-arrange to make k the subject
$$k = \frac{35}{60} = \frac{7}{12}$$

Step 3: Put the value of k found into the equation
$$y = kx$$
$$y = \frac{7}{12}x$$

First, we will find p by using $x = 36$ when $y = p$:
We plug the values of x and y from the column where p is
$$x = 36 \text{ when } y = p$$
$$y = \frac{7}{12}x$$
$$p = \frac{7}{12}(36)$$
$$p = 21$$

Next, we will find q by using $x = q$ when $y = 17.5$:
Plug the values of x and y from the column where q is
$$x = q \text{ when } y = 17.5$$
$$y = \frac{7}{12}x$$
$$17.5 = \frac{7}{12}q$$
$$q = 17.5 \times \frac{12}{7} = 30$$

Hence, the value of **p is 21 and q is 30**

Grade 7/8 Difficulty: Given Two Equations (use algebraic substitution)

These questions involve doing what we already know how to do twice and then using algebraic substitution to eliminate the variable we don't want. This is the same principle as solving simultaneous equations such as
$$x^2 + y^2 = 5$$
$$y = 2x - 4$$

We plug the second equation into the first to eliminate the variable y
$$x^2 + (2x-4)^2 = 5$$
etc

Example 1

y is inversely proportional to d^2. When $d = 10, y = 4$
d is inversely proportional to x^2. When $x = 2, d = 24$
Find a formula for y in terms of x
Give your answer in its simplest form

First, we will calculate two equations : y in terms of d left column) **and** d in terms of x (right column)

Using the info: y is inversely proportional to d^2.

Step 1 : Use the template
y is indirectly proportional to d^2
means
$$y = \frac{k}{d^2}$$

Step 2: Use the first bit of information given to find k
$$y = 4, d = 10$$

We plug these values in our equation to find k
$$y = \frac{k}{d^2}$$
$$4 = \frac{k}{10^2}$$
$$k = 4 \cdot 100 = 400$$

Step 3: Put the value of k found into the equation
$$y = \frac{400}{d^2}$$

Using the info: d is inversely proportional to x^2

Step 1 : Use the template
d is indirectly proportional to x^2
means
$$d = \frac{k}{x^2}$$

Step 2: Use the first bit of information given to find k
$$d = 24, x = 2$$

We plug these values in our equation to find k
$$d = \frac{k}{x^2}$$
$$24 = \frac{k}{2^2}$$
$$k = 24 \cdot 4 = 96$$

Step 3: Put the value of k found into the equation
$$d = \frac{96}{x^2}$$

Next, we plug in the formula for d (on the right) in the formula for y (on the left)
$$y = \frac{400}{d^2} = \frac{400}{\left(\frac{96}{x^2}\right)^2} = \frac{400}{\left(\frac{9216}{x^4}\right)} = \frac{25}{576}x^4$$

Example 2

y is directly proportional to the square root of t
$y = 15$ when $t = 9$
t is inversely proportional to the cube of x
$t = 8$ when $x = 2$
Find a formula for y in terms of x
Give your answer in its simplest form.

First, we will calculate y in terms of t
Step 1 : Use the template
y is directly proportional to $\sqrt{t}$
means
$$y = k\sqrt{t}$$

Step 2: Use the first bit of information given to find k
$$y = 15, t = 9$$

We plug these values in our equation to find k
$$y = k\sqrt{t}$$
$$15 = k\sqrt{9} = k(3)$$
$$k = \frac{15}{3} = 5$$

Step 3: Put the value of k found into the equation
$$y = k\sqrt{t} = 5\sqrt{t}$$

Now, we can find t in terms of x
Step 1 : Use the template
t is indirectly proportional to x^3
means
$$t = \frac{k}{x^3}$$

Step 2: Use the first bit of information given to find k
$$t = 8, x = 2$$

Now, we plug this value to find k
$$8 = \frac{k}{2^3}$$
$$k = 8 \times 8 = 64$$

Step 3: Put the value of k found into the equation
$$t = \frac{64}{x^3}$$

We have
$$y = 5\sqrt{t} \text{ and } t = \frac{64}{x^3}$$
$$y = 5\sqrt{\frac{64}{x^3}}$$
$$y = 5\sqrt{\frac{64}{x^3}}$$
$$y = \frac{40}{\sqrt{x^3}}$$

Grade 7/8 Difficulty: Given variable relationship instead of numbers

Example 1

p is inversely proportion to q^2
When q is doubled, what happens to the value of p?

$$p = \frac{k}{q^2}$$

Since the value is getting doubled, we multiply by 2
$$p = \frac{k}{(2q)^2}$$
$$p = \frac{k}{4q^2}$$
$$p = 0.25\frac{k}{q^2}$$

Hence we are multiplying by 0.25. This means that p is getting quartered **or** reduced by 75%

Note : This is a longer explanation to better understand what we are doing above:

Step 1: Recall the template, and notice that here:

p is inversely proportional to the q^2
means
$$p = \frac{k}{q^2}$$

Step 2: Find the initial value
Initially, the value of q is q
Hence, we can find the initial value
$$\text{Initial value} = \frac{k}{q^2}$$

Step 3: Find the final value
Doubling means multiplying by 2.
Hence, doubling q means we get $2q$

We plug this value in our equation to find the value after the increase
$$\text{Final value} = \frac{k}{(2q)^2} = \frac{k}{4q^2}$$

Step 3: Recall the formula for percentage increase and then substitute the values we know
Percentage increase
$$= \frac{\text{Final value} - \text{Initial value}}{\text{Initial value}} \times 100$$
$$= \frac{\frac{k}{4q^2} - \frac{k}{q^2}}{\frac{k}{q^2}} \times 100$$

Factor the numerator
$$= \frac{\frac{k}{q^2}\left(\frac{1}{4} - 1\right)}{\frac{k}{q^2}} \times 100$$

Cancel the common factor
$$= \left(\frac{1}{4} - 1\right) \times 100 = -75\%$$

Hence the percentage decrease is 75%

Example 2

A pendulum of length L cm has time period T seconds
T is directly proportional to the square root of L
The length of the pendulum is increased by 40%.
Work out the percentage increase in the time period

$$T = k\sqrt{L}$$

Since there is an increase by 40%, we multiply by 1.4
$$T = k\sqrt{1.4L}$$
$$T = k\sqrt{1.4}\sqrt{L}$$
$$T = 1.183k\sqrt{L}$$

Hence we are multiplying by 1.183. This means there is an increase of 0.183, which is 18.3%

Note : This is a longer explanation to better understand what we are doing above:

Step 1: Recall the template, and notice that here:

T is directly proportional to the square root of L
T is directly proportional to the $\sqrt{L}$
means
$$T = k\sqrt{L}$$

Step 2: Find the initial time period
Initially, the length is L
Hence, we can find the initial time period
$$\text{Initial time period} = k\sqrt{L}$$

Step 3: Find the final time period

Increasing by 40% means we are multiplying by 1.4.
Hence, increasing length by 40% means we get $1.4L$

We plug this value in our equation to find the time period after the increase
$$\text{Final time period} = k\sqrt{1.4L}$$

Step 3: Recall the formula for percentage increase and then substitute the values we know
$$\text{Percentage increase} = \frac{\text{Final time period} - \text{Initial time period}}{\text{Initial time peirod}} \times 100$$
$$= \frac{k\sqrt{1.4L} - k\sqrt{L}}{k\sqrt{L}} \times 100$$

Factor the numerator
$$= \frac{k\sqrt{L}\left(\sqrt{1.4} - 1\right)}{k\sqrt{L}} \times 100$$

Cancel the common factor
$$= \left(\sqrt{1.4} - 1\right) \times 100 \approx 18.3$$

Hence the percentage increase in time period is approximately 18.3%

Grade 9 Difficulty

Example 1

y is inversely proportional to the square root of x.
x is directly proportional to the cube of t. Given that $y = 8$ when $t = 25$. Find the exact value of t when y=27

$$y = \frac{k}{\sqrt{x}} \text{ and } x = kt^3$$

First of all, let's understand how this is different to usual:
We are not given a pair of values for y and x
We are also not given a pair of values for x and t

Instead, we are given values for y and t, but they are not connected in one equation. They are connected between TWO equations.
This means we can't find the value of k for each equation, unlike we could for the top row of examples above. k is always different from one equation to another as you have also seen above.

Let's use other variables a and b in to show we know the constants are different
$$y = \frac{a}{\sqrt{x}} \text{ and } x = bt^3$$

We can substitute x into the y equation
$$y = \frac{a}{\sqrt{bt^3}}$$

Split the root up in the denominator
$$y = \frac{a}{\sqrt{b}\sqrt{t^3}}$$

This is now the hard part
We can relabel the constant part since it just stands for a constant anyway. Hence we label the constant part $\frac{a}{\sqrt{b}}$ as k
$$y = \frac{k}{\sqrt{t^3}}$$

We are given that $y = 8, t = 25$,,which we can use to find k
$$8 = \frac{k}{\sqrt{25^3}}$$
$$8 = \frac{k}{125}$$
$$k = 1000$$

So we have
$$y = \frac{1000}{\sqrt{t^3}}$$

We are given $y = 27$
$$27 = \frac{1000}{\sqrt{t^3}}$$
$$27\sqrt{t^3} = 1000$$
$$\sqrt{t^3} = \frac{1000}{27}$$
$$t^3 = \left(\frac{1000}{27}\right)^2$$
$$t^3 = \left(\frac{1000}{27}\right)^{\frac{2}{3}}$$